

# Steady-state many-body entanglement of hot reactive fermions

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Entanglement is typically created via systematic intervention in the time evolution of an initially unentangled state, which can be achieved by coherent control, carefully tailored non-demolition measurements, or dissipation in the presence of properly engineered reservoirs. In this paper we show that two-component Fermi gases at  $\sim \mu\text{K}$  temperatures naturally evolve, in the presence of *reactive* two-body collisions, into states with highly entangled (Dicke-type) spin wavefunctions. The entanglement is a steady-state property that emerges—without any intervention—from uncorrelated initial states, and could be used to improve the accuracy of spectroscopy in experiments with fermionic alkaline earth atoms or fermionic groundstate molecules.

Many-body entangled states are known to be useful for quantum computing, quantum teleportation and cryptography protocols [1], and precision metrology [2]. With these applications as motivation, the physics community has invested tremendous effort in preparing, stabilizing, and measuring entangled systems. Much of this effort has relied on coherent (Hamiltonian) dynamics to arrive at entangled states starting from less exotic states with only classical correlations. However, these approaches typically suffer from the necessity to either carefully engineer interactions between particles or to prepare extremely pure and specific initial states (or both). A bottom up implementation of coherent control has yielded entangled states of up to 14 atoms with relatively high fidelity [3] (in ion experiments), and a top down approach has yielded weakly entangled states in a Bose Einstein condensate of  $\sim 10^4$  neutral atoms [4]. A promising alternative to coherent control is the collective-nondemolition measurement of some observable with a finite variance in an initially classical state. Such approaches have been used to generate entanglement (in the form of spin squeezing) amongst as many as  $10^6$  cold thermal atoms [5]. However, collective and coherence preserving measurements are generically difficult to make, and the induced non-classical correlations are typically weak.

In this paper, we show that two-component non-degenerate fermionic gases can be driven by reactive *s*-wave two-body collisions into steady-state spin configurations that, for a given value of the saturated particle number, are pure and highly entangled. The entanglement comes in the form of Dicke states [6], in which the spin-wavefunction is fully symmetric under interchange of the particles (with the burden of fermionic antisymmetry being taken up entirely by the motional degrees of freedom). Such states have been sought in experiments with ultra-cold bosons for use in Heisenberg-limited phase measurements [7], however these approaches typically suffer from the necessity to reach extremely cold temperatures (for the validity of a two-mode approximation in a double-well potential) or to employ Feshbach resonances [8] (to enhance spin exchange interactions for two-component Bose Einstein condensates). The only requirements to

achieve such entanglement in the steady-state of lossy non-degenerate fermions are to have an  $\text{SU}(2)$  invariant single-particle Hamiltonian (in the pseudo-spin degrees of freedom) and a significant separation of timescales between *s*-wave and *p*-wave collisions, with the second requirement typically being satisfied for temperatures in or below the  $\mu\text{K}$  range.

Because the desired property (i.e. the Dicke type of spin-entanglement) persists in the steady-state of dissipative dynamics, we do not rely on the highly controlled coherent manipulation that is typical of spin-squeezing experiments with bosons. Driven, dissipative preparation of nontrivial steady-states has been considered before in the context of many-body atomic systems [9–11], and has been achieved recently in [12, 13]. In contrast to these examples, the mechanism described here is intrinsic and generic to a variety of interesting and experimentally relevant systems, such as fermionic alkaline-earth atoms (AEAs) and fermionic dipolar molecules, and does not require any special engineering of the system-reservoir coupling. After presenting calculations in support of our claims, we discuss the possible realization of such steady states in an experiment. In particular, we will propose a simple proof of principle experiment in which the steady-state entanglement can be revealed via Ramsey spectroscopy of the  $^1S_0$  to  $^3P_0$  clock transition of an AEA [14]. In this case, we will see that the interferometric precision stays relatively constant even as most of the particles are lost (all but  $\sim \sqrt{N}$  in the long time limit), signaling the development of quantum correlations and the pursuant violation of the standard quantum-limit. The total loss of precision (due to loss of particles) exactly cancels the gain due to the growth of quantum correlations. However, a persistent precision under loss of particles can provide enhanced spectroscopic accuracy; in particular, the steady-state under consideration is largely devoid of mean-field clock shifts.

Our description of spin- $\frac{1}{2}$  fermions with two-body reactive collisions relies on the formalism detailed in Refs. [15–17], generalized for fermions, where we assume the temperature to be sufficiently low that losses are dominantly in the *s*-wave channel. As in Ref. [15], large ki-

netic energy of fermions in the outgoing channels (which for reactive molecules can correspond to temperatures in the 10K range) guarantees they will be rapidly lost from any typical atom trap, justifying a Born Markov approximation. Given a density matrix  $\varrho$  for the system (fermions, Hilbert space  $\mathcal{S}$ ) plus reservoir (outgoing channels of the inelastic collisions, Hilbert space  $\mathcal{R}$ ), the Born-Markov approximation leads to a master equation for the system reduced density matrix  $\rho = \text{Tr}_{\mathcal{R}}[\varrho]$  [16]:

$$\hbar\dot{\rho} = i[\rho, \mathcal{H}] - \frac{\kappa}{2} \int d^3\mathbf{r} (\mathcal{J}^\dagger \mathcal{J} \rho + \rho \mathcal{J}^\dagger \mathcal{J} - 2\mathcal{J} \rho \mathcal{J}^\dagger). \quad (1)$$

The system Hamiltonian  $\mathcal{H} = \mathcal{H}_0 + g \int d^3\mathbf{r} \mathcal{J}^\dagger \mathcal{J}$  is composed of an unspecified single-particle term  $\mathcal{H}_0$  and an interaction term with coupling constant  $g = 4\pi\hbar^2 a_R/m$ , where  $m$  is the particle mass and  $a = a_R + ia_I$  ( $a_I < 0$ ) is the complex  $s$ -wave scattering length. The jump operators are defined by  $\mathcal{J}(\mathbf{r}) = \psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})$  (their explicit  $\mathbf{r}$  dependence is suppressed in the integrals above), where  $\psi_\sigma(\mathbf{r})$  annihilates a fermion located at position  $\mathbf{r}$  in internal state  $\sigma \in \{\uparrow, \downarrow\}$ , and  $\kappa = -4\pi\hbar^2 a_I/m$ . Assuming without loss of generality that the initial number of particles  $\mathcal{N}$  is even, the relevant system Hilbert space can be written as a direct sum over spaces with well defined particle number,  $\mathcal{S} = \mathcal{S}^{\mathcal{N}} \oplus \mathcal{S}^{\mathcal{N}-2} \oplus \dots \oplus \mathcal{S}^0$ , between which coherence never develops. Hence, the density matrix can be decomposed into a sum of density matrices in each particle-number sector, any one of which we label by  $\rho^n$  once normalized. Furthermore, any Hilbert space  $\mathcal{S}^n$  can be decomposed into a direct product between motional (m) and spin (s) degrees of freedom,  $\mathcal{S}^n = \mathcal{S}_m^n \otimes \mathcal{S}_s^n$ , and we can define a reduced spin density matrix by  $\rho_s^n = \text{Tr}_{\mathcal{S}_m^n}[\rho^n]$ . For what follows, it will be useful to define a fidelity in a given Dicke state [28] of the spin degrees of freedom of  $n$  particles,  $|S = n/2, S^z\rangle$ , given by the population of  $\rho_s^n$  in the Dicke state

$$\mathcal{F}_{S, S^z} \equiv \langle S, S^z | \rho_s^n | S, S^z \rangle. \quad (2)$$

Here  $S$  and  $S^z$  are quantum numbers for the total spin and its projection along the  $z$ -axis, respectively.

*Two particles.* To make the physics clear in a simple context, we begin by considering two fermions in a single double well potential (which could be formed in an optical super-lattice [18, 19]). We consider a single wavefunction  $\varphi_\alpha(\mathbf{r})$  in each well ( $\alpha \in \{L, R\}$ ), denote the creation operator for a fermion in spin state  $\sigma$  and wavefunction  $\varphi_\alpha$  by  $\psi_{\sigma\alpha}^\dagger$ , and choose an initial state  $\psi_{\uparrow L}^\dagger \psi_{\downarrow R}^\dagger |vac\rangle$  without spin correlations. Within a tight binding model for these two wavefunctions, the Hamiltonian is

$$\mathcal{H} = -J \sum_{\sigma} (\psi_{\sigma L}^\dagger \psi_{\sigma R} + \psi_{\sigma R}^\dagger \psi_{\sigma L}) + U \sum_{\alpha=L,R} \mathcal{J}_\alpha^\dagger \mathcal{J}_\alpha, \quad (3)$$

and the master equation reads

$$\hbar\dot{\rho} = i[\rho, \mathcal{H}] - \frac{\gamma}{2} \sum_{\alpha=L,R} (\mathcal{J}_\alpha^\dagger \mathcal{J}_\alpha \rho + \rho \mathcal{J}_\alpha^\dagger \mathcal{J}_\alpha - 2\mathcal{J}_\alpha \rho \mathcal{J}_\alpha^\dagger). \quad (4)$$

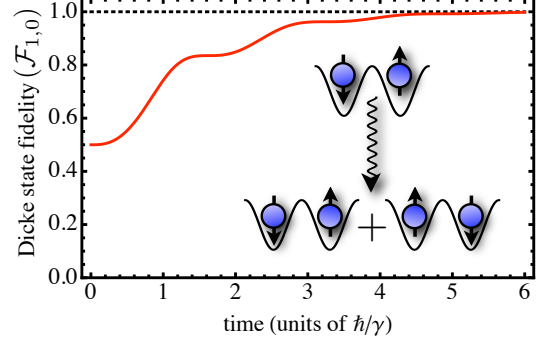


FIG. 1: (Color online). The solid red line is the fidelity of the final density matrix (after post-selection for a non-vacant well) with respect to the  $S_z = 0$  Dicke state, and approaches 1 (black dotted line) at times long compared to  $\gamma^{-1}$ . The oscillations imposed over the exponential decay are due to the inter-well hopping.

Here  $J$  is the inter-well hopping,  $\mathcal{J}_\alpha = \psi_{\uparrow\alpha}^\dagger \psi_{\downarrow\alpha}$ ,  $U = g \int d^3\mathbf{r} |\varphi_\alpha(\mathbf{r})|^4$  is the onsite interaction energy, and  $\gamma = \kappa \int d^3\mathbf{r} |\varphi_\alpha(\mathbf{r})|^4$  is the onsite loss rate. The initial state can be decomposed into an evenly weighted superposition of triplet and singlet  $[(\psi_{\uparrow L}^\dagger \psi_{\downarrow R}^\dagger \pm \psi_{\downarrow L}^\dagger \psi_{\uparrow R}^\dagger) |vac\rangle$ , with plus for the triplet], and the spin wave function of the triplet is the entangled Dicke state  $|1, 0\rangle$ . The triplet,  $|t\rangle$ , having a spin wavefunction that is symmetric under exchange, has an orbital wavefunction that is antisymmetric under exchange, and therefore it is “dark” to  $s$ -wave losses, by which we mean simply that  $\mathcal{J}_L |t\rangle = \mathcal{J}_R |t\rangle = 0$ . It also happens to be an eigenstate of  $\mathcal{H}$ , and so it is stationary under propagation of the master equation (4). On the other hand, there are no dark eigenstates in the singlet sector, and as a result  $\rho_s^2$  is pure at long times and satisfies  $\mathcal{F}_{1,0} = 1$ . In other words the steady state of our system, when restricted to the subspace with two particles, is the entangled Dicke state  $|1, 0\rangle$ . It should be noted that there is also a 50% probability of obtaining the vacuum, and hence in an array of double wells the entanglement fidelity is only unity after post-selection of the non-vacant wells. In this simple example we see an important general feature of the physics we will discuss, that even purely *local* (intra-well) dissipation, when coexisting with Hamiltonian dynamics that delocalizes the particles, generates *non-local* (inter-well) spin correlations in the steady state.

*Many particles.* Solving Eq. (1) exactly for initial states with  $\mathcal{N} > 2$  initial particles quickly becomes impossible, but strong statements can nevertheless be made regarding the steady state. It is crucial to observe that the jump operators only remove spin singlets from the system, which follows from Fermi statistics combined with the even exchange symmetry of the spatial part of any two-particle wave function susceptible to  $s$ -wave scattering. Intuitively, this suggests that losses do not decrease the expectation value of the total spin,

$\mathbf{S} = \frac{1}{2} \int d^3\mathbf{r} \psi_{\sigma}^{\dagger}(\mathbf{r}) \boldsymbol{\tau}_{\sigma\sigma'} \psi_{\sigma'}(\mathbf{r})$  ( $\boldsymbol{\tau}$  being a vector whose components are the Pauli matrices). Mathematically, we say that  $\frac{d}{dt} \langle \mathbf{S} \cdot \mathbf{S} \rangle = \text{Tr}[\rho \mathbf{S} \cdot \mathbf{S}] = 0$ , which can easily be verified in the case when  $\mathcal{H}$  is SU(2) invariant by checking that  $[\mathbf{S} \cdot \mathbf{S}, \mathcal{J}(\mathbf{r})] = 0$ . A stronger consequence of the commutation of all  $\mathcal{J}(\mathbf{r})$  with  $\mathbf{S} \cdot \mathbf{S}$  is that population in any sector of total spin,  $\mathcal{P}_S$ , is also conserved. Because any state with well defined total spin  $S$  must have  $\langle \hat{N} \rangle \geq 2S$  particles (where  $\hat{N} = \int d^3\mathbf{r} \psi_{\sigma}^{\dagger} \psi_{\sigma}$  is the total number operator), an immediate consequence is that the loss of particles can only yield the vacuum deterministically at long times if the initial state is a total spin-singlet. For an uncorrelated spin state, such as a non-degenerate thermal distribution of  $\mathcal{N}$  fermions in a balanced incoherent mixture of  $\uparrow$  and  $\downarrow$ , it can be shown that [20]

$$N(t) \equiv \text{Tr}[\rho \hat{N}] \geq \sum_S 2S \mathcal{P}_S = \frac{\pi^{1/2} \Gamma[\frac{\mathcal{N}}{2} + 1]}{\Gamma[\frac{\mathcal{N}}{2} + \frac{1}{2}]} - 1, \quad (5)$$

which places a lower bound on the steady-state expectation value for the number of particles  $N(t)$ . This expectation value determines the particle number in a typical steady-state configuration, and is achieved (on average) *without* any post selection, but variations of the steady-state particle number will occur from shot to shot. Taking Stirling's approximation for large  $\mathcal{N}$  yields an approximate bound  $N(t) \gtrsim \sqrt{\pi \mathcal{N}/2}$ . For the chosen restriction on the initial state, the validity of Eq. (5) depends only on the SU(2) invariance of  $\mathcal{H}$ , and not on its precise form. Whether the bound (5) is saturated in the steady-state, however, is an important and delicate issue; an affirmative answer guarantees that all of the  $\rho_s^n$  describe pure Dicke states in the steady-state. Demonstrating that this bound is indeed saturated in certain experimentally relevant situations, namely a 1D harmonic trap and a 1D Hubbard chain (optical lattice), is a central technical result of this paper.

Saturation of the bound in Eq. (5) is guaranteed if, for any fixed value of  $n$  and  $S_z$ , the pure density matrix  $|n/2, S_z\rangle\langle n/2, S_z|$  is the *unique* steady-state reduced spin density matrix. This uniqueness, in turn, is equivalent to requiring that any dark-state with quantum numbers  $n$  and  $S_z$  has a well defined spin wavefunction given by the Dicke state  $|n/2, S_z\rangle$ . In the supplement we prove this to be true for a 1D harmonic oscillator potential, and we have verified it numerically for a 1D Hubbard chain (see below). It is worth noting at this point that, while the equivalence of dark-states with the Dicke states is intuitive, there are natural Hamiltonians for which this intuition is incorrect. In particular, all Hamiltonians in  $D > 1$  that are separable in cartesian coordinates do have dark-states with  $\mathcal{F}_{S,S_z} < 1$ .

In order to verify the above statements numerically, we have performed quantum trajectories simulations for an 8-site Hubbard chain with open boundary conditions,

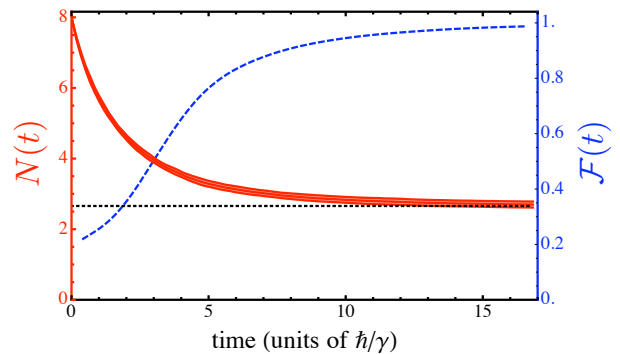


FIG. 2: (Color online). Calculation of particle number  $[N(t)$ , solid red line] and average Dicke state fidelity  $[F(t)$ , dashed blue line] for an 8-site Hubbard chain via quantum trajectories. For the number, the shaded region is an estimate of the statistical error from sampling of a finite number of trajectories. The black dotted line is the analytic bound in Eq. (5).

an initial filling of one particle per site, and zero polarization ( $\mathcal{N} = 8$  and  $S_z = 0$ ). In general we have lower-densities in mind for any experimental application, but using one particle per site allows us to stretch the numerics to the largest  $\mathcal{N}$  possible. In Fig. 2 we show the calculated particle number and average Dicke state fidelity,  $\mathcal{F}(t) = \frac{1}{4} \sum_{S=1}^4 \mathcal{F}_{S,0}$ , and one can see that the former saturates the bound Eq. (5) while the latter approach unity at long times. For this calculation we solve for  $\mathcal{O}(10^4)$  trajectories with no approximation.

*Experimental realization.* Dicke states are known to be useful for a variety of quantum information protocols, including but not limited to quantum secret sharing [21], teleportation [22], and sub shot-noise limited precision spectroscopy [7]. Here we give a brief description of how the proposed Dicke state preparation could be used in precision spectroscopy of the clock transition in alkaline-earth atoms. For a fixed interrogation time, spectroscopy on  $\mathcal{N}$  uncorrelated atoms has a phase sensitivity  $\delta\varphi \gtrsim 1/\sqrt{\mathcal{N}}$ , a bound known as the standard quantum limit (SQL). This bound can be understood as the minimum tipping angle needed to cause a coherent spin state (CSS) to have an uncertainty cone that precludes its initial position [23] (to one standard deviation). On the other hand, spectroscopy on a Dicke state of  $\mathcal{N}$  particles with spin  $S_z = 0$  has the potential to reach the Heisenberg limit (HL) of phase sensitivity,  $\delta\varphi \sim 1/\mathcal{N}$  [7, 23]. It is important to realize that the production of Dicke states with  $\sqrt{\mathcal{N}}$  fermions via two-particle loss does not actually enhance the phase sensitivity relative to the initial state with  $\mathcal{N}$  fermions; the enhancement in phase sensitivity between the SQL and HL exactly compensates the reduction of particle number. However, the reduced particle number in the Dicke state and darkness to *real* s-wave interactions (which if present generate clock shifts), can render the accuracy of the final Dicke state superior

to that of the initial  $\mathcal{N}$  fermion uncorrelated state.

Rather than allowing losses amongst a macroscopic sample of atoms, for which the approach to the steady state could be quite slow, we imagine an array of  $\mathcal{T}$  1D tubes created by a 2D optical lattice. Although there will be variations in the atom number from tube to tube, for simplicity we take each tube to have exactly  $\mathcal{N}$  fermionic AEAs in the  $^1S_0$  electronic state and  $I^z = I$  nuclear-spin state, denoted  $|^1S_0, I\rangle$ . For the analysis in this paper to be valid, the temperature should be small compared to the vibrational level-spacing in the transverse tube direction, and also low enough that only the harmonic part of the trapping potential along the tube axis is sampled by the atoms. A  $\pi/2$ -pulse on the spin degrees of freedom [ $|^1S_0, I\rangle \rightarrow \frac{1}{\sqrt{2}}(|^1S_0, I\rangle + |^1S_0, I-1\rangle)$ ], followed by single particle dephasing [29], generates a statistical mixture of the two spin states ( $I^z = I$  and  $I^z = I-1$ ). Losses can be initiated by applying a  $\pi$ -pulse on the clock transition ( $|^1S_0, I^z\rangle \rightarrow |^3P_0, I^z\rangle$ ). We estimate that this  $\pi$ -pulse [30] can be achieved on the  $\lesssim 100\mu\text{s}$  timescale without exciting transverse excitations in the tubes (which, if present, violate the assumption of a 1D geometry and destroy the uniqueness of the steady-state). Thus the transfer into  $^3P_0$  is sufficiently fast that it can be considered instantaneous on the initial timescale of reactive collisions—which, based on universal considerations for a Lieb-Liniger gas, we estimate to be  $\gtrsim 1\text{ms}$  for experimentally relevant 1D densities [15]—such that it suddenly initiates strong 2-body  $s$ -wave losses.

The steady state of the system is a statistical mixture of Dicke states in the different tubes, each having some value of  $\mathcal{D}_j$  particles (centered around  $\mathcal{D}_0 \approx \sqrt{\mathcal{N}}$ ) and spin projection  $S_j^z$  (centered around zero). Spin selective transfer of  $|^3P_0, I-1\rangle$  into  $|^1S_0, I\rangle$  maps the spin degree of freedom onto the clock states, leaving a spin-polarized sample, and Ramsey spectroscopy on the clock transition can then be performed [23]. Despite the fluctuation of both  $\mathcal{D}_j$  and  $S_j^z$  from one tube to another, it can be shown (see the supplement) that the minimum resolvable rotation angle in a Ramsey experiment scales as

$$\delta\varphi_{\min} \sim 1/\mathcal{D}_0\sqrt{\mathcal{T}}. \quad (6)$$

This result can be interpreted as the existence of Heisenberg limited sensitivity for each tube, which is then combined between tubes in a statistically independent manner (hence the  $1/\sqrt{\mathcal{T}}$ ). In order to utilize this phase sensitivity, the initial value of  $S^z = \sum_j S_j^z$  for the entire ensemble must be accurately known. Because  $S^z$  is conserved by the losses, it can be measured before transfer to the  $^3P_0$ , and hence the measurement does not need to preserve any inter-particle correlations (since these develop during the losses). Accurate measurements of this type and precision for  $\sim 100$  atoms in an optical cavity have recently been demonstrated [24].

The primary limitations on the final state fidelity achievable in experiments is likely to be a combination

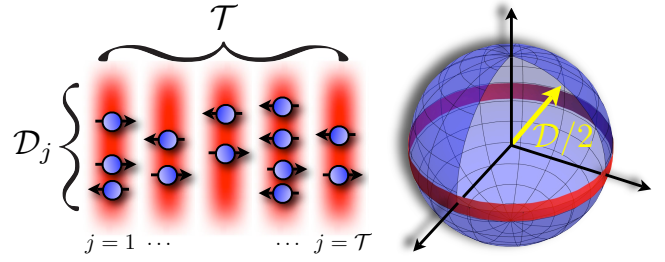


FIG. 3: (Color online). (a) An array of  $\mathcal{T}$  1D tubes, each having  $\mathcal{D}_j$  atoms in a Dicke state. (b) Bloch sphere representation of a Dicke state in a particular tube.

of finite  $p$ -wave losses (which the Dicke states are not dark to) and magnetic field gradients. At sub  $\mu\text{K}$  temperatures, the  $s$ -wave losses in a spin mixture of  $^{87}\text{Sr}$  are expected to be about an order of magnitude faster than the  $p$ -wave losses [25]. For reactive molecules (or  $^{171}\text{Yb}$ ), where the inelastic collisions are expected to more fully saturate the unitarity bound [26], this separation of rates will most likely be even larger. Magnetic field gradients couple sectors of different total  $S$ , all of which are separated from the Dicke manifold by a gap for finite systems and nonzero  $a_R$ , so in principle their adverse effects can be suppressed to first order [27]. Furthermore, if the two components of the Fermi gas are two nuclear spin states of an AEA, they will be extremely insensitive to magnetic field gradients: We estimate that typical gradients ( $1\text{mG/cm}$ ) will cause spin dephasing on a 100s timescale for a linear system size of  $100\mu\text{m}$ . This timescale is several orders of magnitude longer than the initial two-body loss rate in tightly confined 1D tubes, which we estimate to be on the order of 10ms for  $^{87}\text{Sr}$  (assuming a  $50E_R$  2D lattice and scaling the density dependent loss rate from Ref. [25]), and even faster for  $^{171}\text{Yb}$  [26]. A more quantitative analysis of the effects of both magnetic field imperfections and finite  $p$ -wave losses requires numerical simulations beyond the scope of this work, and is left for future study.

*Conclusions.* In this paper we have demonstrated that fairly unrestrictive initial conditions, without intervention or engineering and in the presence of reactive two-body collisions, are sufficient to generate steady-state spin entanglement between non-degenerate fermions. These reactive collisions, which occur both in optically excited alkaline earth atoms and many dipolar molecules (e.g. KRb), are typically viewed as an impediment to interesting physics, but clearly this need not be the case. We expect this physics to enable the distillation of Dicke states from initially uncorrelated fermionic atoms and molecules, hence extending the scope of a variety of experimental progress made in the spin squeezing of bosons.

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- [28] The Dicke state of  $n$  spins having total  $z$  projection of spin  $S^z$  is obtained by acting on the state with maximal  $z$ -projection of total spin  $(n/2 - S^z)$  times with the total spin lowering operator  $S^-$ .
- [29] Dephasing of nuclear spins is typically extremely slow, but can be briefly enhanced with large magnetic field gradients or by scattering photons on a cycling transition.
- [30] This transfer needs to be of high fidelity, since any atoms remaining in the  $^1S_0$  state can undergo  $s$ -wave collisions with  $^3P_0$  atoms that *do* reduce the system spin. In principle, the purity of this transfer could be enhanced by using an optical pumping scheme.

### Uniqueness of the steady state

Here we show that for a 1D harmonic oscillator, in a particular sector of Hilbert space  $\mathcal{H}^n$  and for a particular value of  $S^z$ , the unique steady-state reduced spin density matrix is given by

$$\rho_s^n = |n/2, S^z\rangle\langle n/2, S^z|. \quad (S1)$$

The extension of what follows to the 1D Hubbard chain is fairly straightforward, and will be described in more detail in future work. As discussed in the text, it is sufficient to prove that all dark eigenstates of the non-interacting Hamiltonian

$$\mathcal{H}_{HO} = \int dx \psi_\sigma^\dagger(x) \left( \frac{\partial_x^2}{2m} + \frac{1}{2} m \omega^2 x^2 \right) \psi_\sigma(x) \quad (S2)$$

have a maximally symmetric spin wavefunction. To understand the properties of its dark eigenstates under particle exchange, we will actually work in first quantization writing an eigenstate for  $\mathcal{N}$  particles as

$$\Psi = \sum_{\vec{\sigma}} \mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_{\mathcal{N}}) |\vec{\sigma}\rangle. \quad (S3)$$

Here the  $j^{\text{th}}$  component of the vector  $\vec{\sigma}$ ,  $\sigma_j \in \{\uparrow, \downarrow\}$ , labels the spin orientation of the  $j^{\text{th}}$  particle (along some arbitrary quantization axis, which we'll call  $z$ ), and the total spin wavefunction in any term of the sum is

$$|\vec{\sigma}\rangle = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \dots \otimes |\sigma_{\mathcal{N}}\rangle. \quad (S4)$$

The sum over  $\vec{\sigma}$  should be understood as independent summations over each index

$$\sum_{\vec{\sigma}} = \sum_{\sigma_1} \sum_{\sigma_2} \dots \sum_{\sigma_{\mathcal{N}}}, \quad (S5)$$

the coefficients  $\mathcal{A}_{\vec{\sigma}}$  are arbitrary, and  $\Phi_{\vec{\sigma}}$  is a normalized orbital wavefunction for the  $\mathcal{N}$  particles. Dark states of  $s$ -wave losses have zero expectation value in the interaction operator

$$\mathcal{U} = \sum_{m < n} \mathcal{U}_{mn} = g \sum_{m < n} \delta(r_m - r_n), \quad (\text{S6})$$

and this expectation value can be evaluated as

$$\begin{aligned} U &= \int \mathcal{D}r \Psi^* \mathcal{U} \Psi \\ &= \sum_{\vec{\sigma}} |\mathcal{A}_{\vec{\sigma}}|^2 \int \mathcal{D}r \Phi_{\vec{\sigma}}^* \mathcal{U} \Phi_{\vec{\sigma}}, \end{aligned} \quad (\text{S7})$$

with  $\mathcal{D}r \equiv \prod_j dr_j$ . The last equality holds because the interaction is spin-independent. It is crucial to realize that the operator  $\mathcal{U}$  is positive-semidefinite, which means that satisfying  $U = 0$  actually implies the stricter constraint

$$\int \mathcal{D}r \Phi_{\vec{\sigma}}^* \mathcal{U} \Phi_{\vec{\sigma}} = 0 \quad \forall \vec{\sigma}. \quad (\text{S8})$$

In addition to the operator  $\mathcal{U}$  being positive-semidefinite, the constituent pairwise interaction operators are as well. Hence, the condition  $\int \mathcal{D}r \Phi_{\vec{\sigma}}^* \mathcal{U} \Phi_{\vec{\sigma}} = 0$  actually implies that

$$\int \mathcal{D}r \Phi_{\vec{\sigma}}^* \mathcal{U}_{mn} \Phi_{\vec{\sigma}} = 0 \quad \forall m \neq n. \quad (\text{S9})$$

The above set of equalities can now be used to pin down properties regarding the exchange symmetry of the wave functions  $\Phi_{\vec{\sigma}}$ .

#### *Implications for the exchange symmetry of the orbital wave functions*

Let's choose two particles, say particle 1 and particle 2, and define relative and center-of-mass coordinates for them as  $r = r_1 - r_2$  and  $R = (r_1 + r_2)/2$ . We can then expand the wavefunction  $\Phi_{\vec{\sigma}}$  in a way that makes the symmetrization with respect to exchange of particles 1 and 2 explicit:

$$\Phi_{\vec{\sigma}} = \sum_{\beta s} \mathcal{B}_{\vec{\sigma}\beta s} \times \varphi_{\alpha(\beta, s)}(r) \varphi_{\beta}(R) \psi_s(r_3, \dots, r_N). \quad (\text{S10})$$

In the above  $\varphi_{\alpha}$  are harmonic oscillator wavefunctions, the  $\psi_s$  are a complete and orthonormal set of eigenfunctions for the remaining  $\mathcal{N} - 2$  particles, and the notation  $\alpha(\beta, s)$  implies that the relative wavefunction of particles 1 and 2 is uniquely determined by  $\beta$  and  $s$ . This point is crucial, and relies on the observation that for  $\Psi$  to be an eigenstate, when expanded in terms of eigenstates  $\Phi_{\vec{\sigma}}$  all of the eigenstates must have the same eigenvalue. Hence the energies of states  $\alpha$ ,  $\beta$ , and  $s$  are constrained to add

to some fixed value. From now on we'll drop this explicit dependence. The interaction energy between particles 1 and 2 is given by

$$\begin{aligned} U_{12} &= \int \mathcal{D}r \Phi_{\vec{\sigma}}^* \mathcal{U}_{12} \Phi_{\vec{\sigma}} \\ &= g \sum_{\beta s} \mathcal{B}_{\vec{\sigma}\beta s} \varphi_{\alpha}(0) \times \mathcal{B}_{\vec{\sigma}\beta s}^* \varphi_{\alpha}^*(0) \\ &= \sum_{\beta s} |\mathcal{B}_{\vec{\sigma}\beta s} \varphi_{\alpha}(0)|^2, \end{aligned} \quad (\text{S11})$$

and hence  $U_{12} = 0$  implies that

$$\mathcal{B}_{\vec{\sigma}\beta s} \varphi_{\alpha}(0) = 0 \quad \forall \beta, s \quad (\text{S12})$$

So  $\mathcal{B}_{\vec{\sigma}\beta s}$  must be zero for all even wave functions (all of which are finite at the origin), implying that  $\Phi_{\vec{\sigma}}$  is strictly odd under interchange of particles 1 and 2. By repeating the above argument for two arbitrary particles  $m$  and  $n$ , it is easy to see that  $\Phi_{\vec{\sigma}}$  is strictly odd under interchange of *any* two particles.

#### *Implications for the spin wavefunction*

We can now ask what the antisymmetry of  $\Phi_{\vec{\sigma}}$  implies for the full wave function

$$\Psi = \sum_{\vec{\sigma}} \mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_N) |\vec{\sigma}\rangle. \quad (\text{S13})$$

Under interchange of two arbitrary particles we have  $|\vec{\sigma}\rangle \rightarrow |\vec{\sigma}'\rangle$ , and we obtain the new wave function

$$\begin{aligned} \Psi' &= - \sum_{\vec{\sigma}} \mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_N) |\vec{\sigma}'\rangle \\ &= - \sum_{\vec{\sigma}} \mathcal{A}_{\vec{\sigma}'} \Phi_{\vec{\sigma}'}(r_1, \dots, r_N) |\vec{\sigma}\rangle \\ &= - \Psi \\ &= - \sum_{\vec{\sigma}} \mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_N) |\vec{\sigma}\rangle, \end{aligned} \quad (\text{S14})$$

implying that

$$\mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_N) = \mathcal{A}_{\vec{\sigma}'} \Phi_{\vec{\sigma}'}(r_1, \dots, r_N). \quad (\text{S15})$$

The second equality follows because switching  $\sigma \leftrightarrow \sigma'$  in the summand just changes the order of the terms in the sum, and the third equality follows from the antisymmetry of the total wavefunction under particle exchange. By repeated permutations of various particles, this chain of logic can be used to demonstrate that all of the  $\mathcal{A}_{\vec{\sigma}} \Phi_{\vec{\sigma}}(r_1, \dots, r_N)$  are equal, and hence we have

$$\Psi = \mathcal{A} \Phi(r_1, \dots, r_N) \sum_{\vec{\sigma}} |\vec{\sigma}\rangle. \quad (\text{S16})$$

Now  $\mathcal{A}$  is just some normalization, which is related to the total  $z$  projection of the spin, and it is clear that  $\Psi$  breaks up into the product of a completely antisymmetric orbital wavefunction multiplied by a completely symmetric spin wavefunction.



### Phase sensitivity of the steady-state

In order to estimate the phase measurement sensitivity of an array of 1D tubes in the steady state, we begin by considering just the  $j^{\text{th}}$  tube, with initial particle number  $\mathcal{N}$ , final particle number  $\mathcal{D}_j$  after relaxing to steady state via collisional loss, and initial (and final) spin projection  $S_j^z$ . The initial  $\mathcal{N}$  atoms can most easily be prepared in an incoherent mixture of spin up and spin down by simply allowing a coherent state initially prepared along the  $x$ -direction to undergo single particle dephasing (which could be briefly enhanced via a myriad of methods). For the coherent state, the probability of a given  $S_j^z$  is given by a binomial distribution, which for large  $\mathcal{N}$  is approximated by the continuous probability distribution

$$\mathcal{P}(S_j^z) = \sqrt{\frac{2}{\mathcal{N}\pi}} e^{-2(S_j^z/\sqrt{\mathcal{N}})^2}. \quad (\text{S17})$$

Note that without dephasing into a mixture, a coherent state of fermions does not undergo  $s$ -wave collisions. In the sense that such a distribution is easily prepared experimentally, we take this to be a worst-case scenario; a distribution of  $S_j^z$  more sharply peaked around  $S_j^z = 0$  will enhance the phase sensitivity. The probability distribution of steady-state particle numbers in the  $j^{\text{th}}$  tube, conditioned on a particular value of  $S_j^z$ , is given by

$$\mathcal{P}(\mathcal{D}_j|S_j^z) = \Theta(\mathcal{D}_j - |2S_j^z|) \frac{2\mathcal{D}_j}{\mathcal{N}} e^{-(\mathcal{D}_j/\sqrt{2\mathcal{N}})^2} e^{2(S_j^z/\sqrt{\mathcal{N}})^2}. \quad (\text{S18})$$

For  $S_j^z = 0$ , this distribution is peaked around  $\mathcal{D}_0 \approx \sqrt{\mathcal{N}}$  (giving the expected value of  $n(\infty)$  quoted in the text).

For  $|S_j^z| > 0$ , the step function  $\Theta$  reflects the fact that as particles are lost (remember that  $S_j^z$  is conserved by the losses), the remaining particles are maximally spin polarized once  $\mathcal{D}_j = 2|S_j^z|$ . The second exponential provides the proper normalization,  $\frac{1}{2} \int_0^\infty d\mathcal{D}_j \mathcal{P}(\mathcal{D}_j|S_j^z) = 1$ , where the factor of  $\frac{1}{2}$  comes from converting sums into integrals while respecting our assumption of even particle number.

For this single tube, small rotations about the  $x$ -axis by an angle  $\delta\varphi$  cause a standard deviation in the final distribution of  $S_j^z$  given by [23]

$$\sigma(\delta\varphi, \mathcal{D}_j, S_j^z) \approx \delta\varphi \sqrt{\frac{(\mathcal{D}_j - 2S_j^z)(\mathcal{D}_j + 2S_j^z)}{8}}. \quad (\text{S19})$$

for large  $\mathcal{D}_j$ . For  $S_j^z = 0$  (before the rotation), this demonstrates that a discrepancy in  $S_j^z$  (after the rotation) of order unity is expected for  $\delta\varphi \sim 1/\mathcal{D}_j$ , hence the Heisenberg limited phase-sensitivity within a single tube.

Our estimation of the phase sensitivity for an array of tubes relies only on the assumption that the *initial* (i.e. before the losses) value of  $S^z = \sum_j S_j^z$  is known to within an uncertainty  $\Sigma$ , but does not require any knowledge of  $S_j^z$  in the individual tubes, greatly relaxing the experimental requirements. This uncertainty  $\Sigma$  guarantees that, in principle, rotations causing deviations in  $S^z$  of order  $\Sigma$  can be detected. For  $\mathcal{T}$  tubes with well defined (i.e. measured) total spin projection  $S^z$ , the expected standard deviation in  $S^z$  (denoted  $\sigma_{\text{tot}}$ ) due to a rotation by angle  $\varphi$  about the  $x$ -axis satisfies

$$\sigma_{\text{tot}}^2 = \overbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}^{\mathcal{T} \text{ times}} \prod_j dS_j^z \overbrace{\int_0^{\infty} \dots \int_0^{\infty}}^{\mathcal{T} \text{ times}} \prod_j d\mathcal{D}_j \prod_j \left[ \frac{1}{2} \mathcal{P}(\mathcal{D}_j|S_j^z) \mathcal{P}(S_j^z) \right] \sum_j \sigma(\delta\varphi, \mathcal{D}_j, S_j^z)^2 \delta(S^z - \sum_j S_j^z). \quad (\text{S20})$$

Here  $\delta$  is the Dirac  $\delta$ -function, reflecting the correlations established between the various  $S_j^z$  by the knowledge of  $S^z$ , and the factors of  $\frac{1}{2}$  again come from converting sums into integrals while respecting the assumption of even particle number. If we ignore this  $\delta$ -function constraint, which is valid in the large  $\mathcal{T}$  limit, the integral simplifies greatly to

$$\sigma_{\text{tot}}^2 \approx \frac{\mathcal{T}}{2} \int_{-\infty}^{\infty} dS_j^z \int_0^{\infty} d\mathcal{D}_j \mathcal{P}(\mathcal{D}_j|S_j^z) \mathcal{P}(S_j^z) \sigma(\delta\varphi, \mathcal{D}_j, S_j^z)^2. \quad (\text{S21})$$

Equation (S21) can be evaluated explicitly to reveal

$$4\sigma_{\text{tot}}^2 \approx \mathcal{T}\mathcal{N}\delta\varphi^2 \approx \mathcal{T}\mathcal{D}_0^2\delta\varphi^2. \quad (\text{S22})$$

Setting the total standard deviation to  $\Sigma$  gives a minimum phase sensitivity of

$$\delta\varphi_{\text{min}} \approx \frac{2\Sigma}{\mathcal{D}_0\sqrt{\mathcal{T}}}. \quad (\text{S23})$$

As has been demonstrated recently in Ref. [24],  $\Sigma \sim 1$  is possible for  $\sim 100$  atoms in an optical cavity, as long as measurements that do not preserve coherence between the atoms are acceptable. Such measurements certainly are acceptable before the losses take place, since we require no inter-particle correlations in the initial state (they develop dynamically due to the losses). Therefore, as quoted in the manuscript, we expect a minimum

phase sensitivity of  $\delta\varphi_{\min} \sim \frac{1}{\mathcal{D}_0\sqrt{\mathcal{T}}}$  to be achievable in experiment.